Appendix V to:
A Theoretical Investigation to the Physical Constraints for Light Velocity in Empty Euclidian Space and first Consequences for Long-distance Physics.
Referred equations: (24), (25), (26), (34a), (43), (44), (45), (46), (47), (48), (49), (50), (51)

❶ *Testing combined displacement in the same direction*

Here, the non-moving system equations (24) – (26) are applicable. If we apply a transformation twice, d_1 and d_2 respectively, the end result should be a non-moving system transformation also, with a combined distance d. This will be worked out starting from the time transformation equation (24) applied to the transformed variables then using equations (24), (25):

(V.1) $$(T^*)^* = \frac{\gamma^2 - x^* d_1}{\gamma\sqrt{\gamma^2 - d_1^2}} T^* = \frac{1}{\gamma\sqrt{\gamma^2 - d_1^2}}\left\{\gamma^2 \frac{\gamma^2 - x\, d_2}{\gamma\sqrt{\gamma^2 - d_2^2}} - d_1 \frac{\gamma\,(x - d_2)}{\sqrt{\gamma^2 - d_2^2}}\right\} T = \frac{\gamma^2 + d_1 d_2 - x\,(d_1 + d_2)}{\sqrt{(\gamma^2 - d_1^2)(\gamma^2 - d_2^2)}} T$$

and this should be identical to a transformation with a combined d:

(V.2) $$(T^*)^* \equiv \frac{\gamma^2 - x\, d}{\gamma\sqrt{\gamma^2 - d^2}} T$$

so that, from comparing the terms in the numerators:

(V.3) $$(\gamma^2 + d_1 d_2) \div (d_1 + d_2) = \gamma^2 \div d$$

leading to:

(46) $$d = \frac{\gamma^2 (d_1 + d_2)}{\gamma^2 + d_1 d_2}$$

where indeed, concerning the denominator:

(V.4) $$\gamma\sqrt{\gamma^2 - d^2} = \gamma \frac{\sqrt{\gamma^2 (\gamma^2 + d_1 d_2)^2 - \gamma^4 (d_1 + d_2)^2}}{\gamma^2 + d_1 d_2} = \frac{\gamma^2}{\gamma^2 + d_1 d_2}\sqrt{(\gamma^2 - d_1^2)(\gamma^2 - d_2^2)}$$

❷ *Testing combined displacement in perpendicular directions*

Again applying a non-moving system transformation twice, now in perpendicular directions d_x and d_y, the end result should be a non-moving system transformation with a combined distance d and a combined direction $\vec{x}$. Again, this will be worked out starting from the time transformation, where the first step is in the y-direction – note, that this is the x-direction of the perpendicular system:

(V.5) $$(T^*)^* = \frac{\gamma^2 - y^* d_y}{\gamma\sqrt{\gamma^2 - d_y^2}} T^* = \frac{1}{\gamma\sqrt{\gamma^2 - d_y^2}}\left\{\gamma^2 \frac{\gamma^2 - x\, d_x}{\gamma\sqrt{\gamma^2 - d_x^2}} - y\, d_y\right\} T = \frac{\gamma^2 - x\, d_x - y\, d_y \frac{\sqrt{\gamma^2 - d_x^2}}{\gamma}}{\sqrt{(\gamma^2 - d_x^2)(\gamma^2 - d_y^2)}} T$$

The x and y will be components of the new vector $\vec{x}$:

(V.6) $x = \tilde{x}\cos(\phi) \qquad y = \tilde{x}\sin(\phi)$

where $\tilde{x} = |\vec{x}|$, so that:

(V.7) $$(T^*)^* = \frac{\gamma^2 - \tilde{x}\left\{\cos(\phi)\, d_x + \sin(\phi)\, d_y \frac{\sqrt{\gamma^2 - d_x^2}}{\gamma}\right\}}{\sqrt{(\gamma^2 - d_x^2)(\gamma^2 - d_y^2)}} T$$

And that should be identical to:

(V.8) $$(T^*)^* = T^*(d,\tilde{x};0,T) = \frac{\gamma^2 - \tilde{x}\, d}{\gamma\sqrt{\gamma^2 - d^2}} T$$

Since the constant term γ^2 in the numerator is the same, the denominators (squared) must be identical:

(V.9) $$\gamma^2 (\gamma^2 - d^2) = (\gamma^2 - d_x^2)(\gamma^2 - d_y^2)$$

leading to:

(47) $$d = \sqrt{d_x^2 + d_y^2 - \frac{d_x^2 d_y^2}{\gamma^2}}$$

and the angle ϕ can be found from equating the parts following $\tilde{x}$ in the numerator:

(V.10) $$d = \cos(\phi)\, d_x + \sin(\phi)\, d_y \frac{\sqrt{\gamma^2 - d_x^2}}{\gamma}$$

which can be solved after some math as:

(V.11) $$\cos(\phi) = \frac{d_x}{d} \qquad \sin(\phi) = \frac{d_y \sqrt{\gamma^2 - d_x^2}}{\gamma\, d} \qquad \text{so} \qquad \tan(\phi) = \frac{d_y \sqrt{\gamma^2 - d_x^2}}{\gamma\, d_x}$$

The first part being equation (48).

❸ *Force-free movement in a force-free moving system along the same direction*

Here and in the following case, the moving system equations (34a) and (43) – (45) are applicable. Consider a system moving in the x-direction according to $d(T) = d_\infty (1 - T_{or}/T)$ and an object force-free moving in that system also along the x-direction, coordinates x^*, y^* where $y^* = 0$, so according to:

(V.12) $$x^*(T^*) = x^*_\infty (1 - T^*_x / T^*)$$

which is better worked out when both sides are multiplied by T^*:

(V.13) $$x^* T^* = x^*_\infty (T^* - T^*_x)$$

Applying the moving-system equations (43), (44) to both sides:

(V.14) $$\frac{\gamma\, \{ (x - d_\infty)\, T + d_\infty T_{or} \}}{\sqrt{\gamma^2 - d_\infty^2}} = x^*_\infty \left\{ T^*_{or} + \frac{\gamma^2 - x\, d_\infty}{\gamma \sqrt{\gamma^2 - d_\infty^2}}\, T - \frac{\gamma}{\sqrt{\gamma^2 - d_\infty^2}}\, T_{or} - T^*_x \right\}$$

which after multiplying both sides with $\gamma \sqrt{\gamma^2 - d_\infty^2}$ can be rearranged to:

(V.15) $$(\gamma^2 + x^*_\infty d_\infty)\, x T = \gamma^2 (x^*_\infty + d_\infty)\, T - \gamma\, x^*_\infty \left\{ \gamma T_{or} + \sqrt{\gamma^2 - d_\infty^2}\, (T^*_x - T^*_{or}) \right\} - \gamma^2 d_\infty T_{or}$$

so that also in the observer system it is a force-free movement:

(V.16) $$x T = x_\infty (T - T_x)$$

where

(49) $$x_\infty = \frac{\gamma^2 (x^*_\infty + d_\infty)}{\gamma^2 + x^*_\infty d_\infty}$$

(V.17) $$T_x = T_{or} + \frac{x^*_\infty \sqrt{\gamma^2 - d_\infty^2}}{\gamma\, (x^*_\infty + d_\infty)}\, (T^*_x - T^*_{or})$$

❹ *Force-free movement in a force-free moving system perpendicular to its observed direction*

Now consider a system moving in the x-direction according to $d(T) = d_\infty (1 - T_{or}/T)$ and an object force-free moving in that system along the y-direction, coordinates x^*, y^* where $x^* = 0$, so according to:

(V.18) $$y^*(T^*) = y^*_\infty\, (1 - T^*_y / T^*)$$

Since for the moving object $x^* = 0$, it follows directly from equation (44) that for its x-coordinate $x(T)$ in the observer's system:

(V.19) $$0 = \gamma^2 [\, \{ (x(T) - d_\infty \}\, T + d_\infty T_{or} \,] \quad \Rightarrow \quad x(T) = d_\infty (1 - T_{or}/T)$$

which can be inserted in equation (43) to express its T^* in terms of T:

(V.20) $$T^* = T^*_{or} + \frac{\gamma^2 T - d_\infty^2 (T - T_{or}) - \gamma^2 T_{or}}{\gamma \sqrt{\gamma^2 - d_\infty^2}} = T^*_{or} + \frac{\sqrt{\gamma^2 - d_\infty^2}}{\gamma}\, (T - T_{or})$$

Then, from equation (V.18):

(V.21) $$y^*(T^*)\, T^* = y^*_\infty (T^* - T^*_y) = y^*_\infty \frac{\sqrt{\gamma^2 - d_\infty^2}}{\gamma}\, (T - T_{or}) + y^*_\infty (T^*_{or} - T^*_y)$$

which according to the combination of equations (43) and (45) is equal to $y(T)\, T$ so that:

(V.22) $$y(T) = y^*_\infty \frac{\sqrt{\gamma^2 - d_\infty^2}}{\gamma}\, (1 - T_{or}/T) + y^*_\infty (T^*_{or} - T^*_y)/T$$

For $T^*_{or} = T^*_y$, these $x(T), y(T)$ should describe a trajectory of a force-free moving object passing the origin at time T_{or} with limit distance $\tilde{x}_\infty$ under an angle ϕ with the x-axis:

(V.23) $$\begin{cases} x(T) = \tilde{x}_\infty \cos(\phi)\, (1 - T_{or}/T) \\ y(T) = \tilde{x}_\infty \sin(\phi)\, (1 - T_{or}/T) \end{cases}$$

so that from comparison with the above found $x(T), y(T)$, equations (V.19), (V.22):

(V.24) $$\begin{cases} \tilde{x}_\infty \cos(\phi) = d_\infty \\ \tilde{x}_\infty \sin(\phi) = y^*_\infty \frac{\sqrt{\gamma^2 - d_\infty{}^2}}{\gamma} \end{cases}$$

The top part is equation (51):

(51) $$\cos(\phi) = \frac{d_\infty}{\tilde{x}_\infty}$$

And the squared sum:

(V.25) $$\tilde{x}_\infty{}^2 = d_\infty{}^2 + y^{*2}_\infty \frac{\gamma^2 - d_\infty{}^2}{\gamma^2} = d_\infty{}^2 + y^{*2}_\infty - \frac{d_\infty{}^2 y^{*2}_\infty}{\gamma^2}$$

leads to equation (50).

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