Appendix D to:
A Theoretical Investigation to the Physical Constraints for Light Velocity in Empty Euclidian Space and first Consequences for Long-distance Physics.
Referred equations: (17), (18), (19), (B.19), (B.20)

Derivation of the space transformations for displacement

A coordinate transformation is considered over a constant distance $\vec{d}$ in the x-direction:

(D.1) $$\vec{d} = \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}$$

and for the vectors $\vec{s}$ and $\vec{r}$, photon direction μ and transformation functions X( ) and Y( ) in polar coordinates as in Appendix B. For displacement d, the time transformation was proven to be:

(17) $$T^*(d;x,y,T) = \Theta_d\,(\gamma^2 - x\,d)\,T - \Delta t_d$$

which is inserted into equation (B.19) where $x = s\cos(\sigma)$:

(D.2) $$Y(s,\sigma,T_s) = C_d\,s\sin(\sigma)\frac{T_s}{T_s^*} = \frac{C_d\,s\sin(\sigma)\,T_s}{\Theta_d\,\{\gamma^2 - s\,d\cos(\sigma)\}\,T_s - \Delta t_d}$$

In the limit for $|T_s| \to \infty$, $s \to \gamma$, $\sigma \to \mu$ this becomes:

(D.3) $$Y(\gamma,\mu,\infty) = \frac{C_d\sin(\mu)}{\Theta_d\,\{\gamma - d\cos(\mu)\}}$$

and that value will have a maximum of γ reached for a μ_0 for which $dY/d\mu = 0$:

(D.4) $$0 = \frac{\cos(\mu_0)}{\gamma - d\cos(\mu_0)} - \frac{d\sin(\mu_0)^2}{\{\gamma - d\cos(\mu_0)\}^2}$$

where the constants C_d, Θ_d have been left out; the solution for μ_0 is:

(D.5) $$\cos(\mu_0) = \frac{d}{\gamma} \quad \text{—>} \quad \sin(\mu_0) = \sqrt{1 - \frac{d^2}{\gamma^2}}$$

so that (the solution for the sinus is positive since Y( ) is maximal so positive), from equation (D.3):

(D.6) $$\gamma = \frac{C_d\sqrt{1 - d^2/\gamma^2}}{\Theta_d\,(\gamma - d^2/\gamma)}$$

which implies that C_d can be expressed in terms of Θ_d:

(D.7) $$C_d = \Theta_d\,\gamma\,\sqrt{\gamma^2 - d^2}$$

and the solution for Y( ), equation (D.2), becomes – now written in terms of r, α:

(D.8) $$Y(r,\alpha,T_r) = \frac{\gamma\sqrt{\gamma^2 - d^2}\;r\sin(\alpha)\,T_r}{\{\gamma^2 - r\,d\cos(\alpha)\}\,T_r - \Delta t_d/\Theta_d}$$

which is, in Cartesian coordinates replacing T_r by T and in the notation including the parameter d:

(19) $$y^*(d;x,y,T) = \frac{\gamma\sqrt{\gamma^2 - d^2}\;y\,T}{(\gamma^2 - x\,d)\,T - \Delta t_d/\Theta_d}$$

In the limit for $|T_r| \to \infty$, $r \to \gamma$, $\alpha \to \mu$:

(D.9) $$Y(\gamma,\mu,\infty) = \frac{\gamma\sqrt{\gamma^2 - d^2}\;\sin(\mu)}{\gamma - d\cos(\mu)}$$

For the x-components, it was shown in Appendix B that:

(B.20) $$X(r,\alpha,T_r)\,T_r^* = X(\gamma,\mu,\infty)\,(T_r^* - T_s^*) + X(s,\sigma,T_s)\,T_s^*$$

The special case $\vec{s} = \vec{d}$, $T_s = T_d$ is considered since in that case the full right-hand side is known. Firstly, $X(d,0,T_d) = 0$ since the transformed coordinate is the new origin. Secondly, $X(\gamma,\mu,\infty)$ is known since $X(\gamma,\mu,\infty)^2 + Y(\gamma,\mu,\infty)^2$ has to be equal to γ^2 so that from equation (D.9):

(D.10) $$X(\gamma,\mu,\infty) = \pm\sqrt{\gamma^2 - \frac{\gamma^2(\gamma^2 - d^2)\sin(\mu)^2}{\{\gamma - d\cos(\mu)\}^2}} = +\frac{\gamma\,\{\gamma\cos(\mu) - d\}}{\gamma - d\cos(\mu)}$$

See below why the + sign pertains. Inserting into equation (B.20), together with equation (17) where $x = r\cos(\alpha)$ and for $\vec{s} = \vec{d}$, $T_s = T_d$ in the right-hand side:

(D.11) $$X(r,\alpha,T_r)\,T_r^* = \frac{\gamma\,\{\gamma\cos(\mu) - d\}}{\gamma - d\cos(\mu)}\Big(\Theta_d\,\{\gamma^2 - r\,d\cos(\alpha)\}\,T_r - \Delta t_d - \{\Theta_d\,(\gamma^2 - d^2)\,T_d - \Delta t_d\}\Big)$$

where T_r and T_d are connected through a photon path $(\vec{\gamma} - \vec{r})\, T_r = (\vec{\gamma} - \vec{d})\, T_d$. Taking the x-components of this path equation gives:

(D.12) $\{\gamma \cos(\mu) - r\cos(\alpha)\}\, T_r = \{\gamma \cos(\mu) - d\}\, T_d$

and inserting the resulting T_d in the above equation for $X(r,\alpha,T_r)$:

(D.13) $$X(r,\alpha,T_r)\, T_r^* = \frac{\gamma\, \Theta_d\, T_r}{\gamma - d\cos(\mu)} \big(\{\gamma \cos(\mu) - d\}\{\gamma^2 - r\, d\cos(\alpha)\} - \{\gamma \cos(\mu) - r\cos(\alpha)\}(\gamma^2 - d^2) \big)$$
$$= \gamma^2\, \Theta_d\, \{r\cos(\alpha) - d\}\, T_r$$

so that, replacing T_r by T and again applying equation (17):

(D.14) $$X(r,\alpha,T) = \frac{\gamma^2 \{r\cos(\alpha) - d\}\, T}{\{\gamma^2 - r\, d\cos(\alpha)\}\, T - \Delta t_d/\Theta_d}$$

In Cartesian coordinates and in the notation including the parameter d:

(18) $$x^*(d;x,y,T) = \frac{\gamma^2 (x - d)\, T}{(\gamma^2 - x\, d)\, T - \Delta t_d/\Theta_d}$$

Additional consideration

Since only a displacement is considered, no rotation, the photon's limit locations will be in the same direction in both the original and transformed system. For positive T_0, this is $x^* = x = +\gamma$ for $T \to \infty$ and for negative T_0 $x^* = x = -\gamma$ for $T \to -\infty$. Both cases are consistent with equation (18) so a +-sign in equation (D.10).

The equations for $X(r,\alpha,T)$, $Y(r,\alpha,T)$, $T^*(T,r,\alpha)$ have been derived using special cases like $|T| \to \infty$ and $\vec{s} = \vec{d}$ but should be checked for general validity. Concerning the x-components, equation (B.20) can be written as, substituting $\gamma \cos(\mu^*)$ for $X(\gamma,\mu,\infty)$:

(D.15) $\gamma \cos(\mu^*)\, T_r^* - X(r,\alpha,T_r)\, T_r^* = \gamma \cos(\mu^*)\, T_s^* - X(s,\sigma,T_s)\, T_s^*$

and, with $X(\,)$, $T^*(\,)$ substituted and both sides divided by Θ_d:

(D.16) $[\gamma \cos(\mu^*)\{\gamma^2 - r\, d\cos(\alpha)\} - \gamma^2 \{r\cos(\alpha) - d\}]\, T_r = [\gamma \cos(\mu^*)\{\gamma^2 - s\, d\cos(\sigma)\} - \gamma^2 \{s\cos(\sigma) - d\}]\, T_s$

Combined with the same photon path equation but now for the untransformed system:

(D.17) $[\gamma \cos(\mu) - r\cos(\alpha)]\, T_r = [\gamma \cos(\mu) - s\cos(\sigma)]\, T_s$

the times T_r, T_s can be eliminated:

(D.18) $$[\gamma \cos(\mu^*)\{\gamma^2 - r\, d\cos(\alpha)\} - \gamma^2 \{r\cos(\alpha) - d\}]\,[\gamma \cos(\mu) - s\cos(\sigma)] =$$
$$[\gamma \cos(\mu^*)\{\gamma^2 - s\, d\cos(\sigma)\} - \gamma^2 \{s\cos(\sigma) - d\}]\,[\gamma \cos(\mu) - r\cos(\alpha)]$$

which, after some reshuffling, can be simplified to:

(D.19) $$\cos(\mu^*) = \frac{\gamma \cos(\mu) - d}{\gamma - d\cos(\mu)}$$

independent of any $\vec{r}$, $\vec{s}$, T_r, T_s so generally valid indeed. The corresponding treatment for $Y(\,)$ leads to the equivalent equation:

(D.20) $$\sin(\mu^*) = \frac{\sqrt{\gamma^2 - d^2}\, \sin(\mu)}{\gamma - d\cos(\mu)}$$

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